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2002 J. Phys. A: Math. Gen. 35 343

(http://iopscience.iop.org/0305-4470/35/2/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 343–349 PII: S0305-4470(02)29535-9

The first-integral method to study the Burgers–Korteweg–de Vries equation

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Received 1 October 2001, in final form 29 October 2001
Published 4 January 2002
Online at stacks.iop.org/JPhysA/35/343

Abstract
In this paper, applying the theory of commutative algebra, we propose a new approach which we currently call the first-integral method to study the Burgers–Korteweg–de Vries equation.

PACS numbers: 02.30.Jr., 02.10.Pk, 04.20.Jb, 05.45−a
Mathematics Subject Classification: 13P10, 34C05, 34C20, 35Q53

1. Introduction

During the past three decades, the Burgers equation, Korteweg–de Vries (KdV) equation and Burgers–Korteweg–de Vries equation (Burgers–KdV) have attracted a lot of attention from a rather diverse group of scientists such as physicists and mathematicians, because these three equations not only arise from realistic physical phenomena, but can also be widely applied to many physically significant fields such as plasma physics, fluid dynamics, crystal lattice theory, nonlinear circuit theory and astrophysics [1–10].

Consider the Burgers–KdV equation

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0 \tag{1}$$

where α, β and s are real constants with $\alpha\beta s \neq 0$. Equation (1) is applied as a nonlinear model of the propagation of waves on an elastic tube filled with a viscous fluid [7], the flow of liquids containing gas bubbles [8] and turbulence [9, 10]. It can also be regarded as a combination of the Burgers' equation and KdV equation, since the choices $\alpha \neq 0$, $\beta \neq 0$ and $s = 0$ lead equation (1) to the Burgers' equation

$$u_t + \alpha u u_x + \beta u_{xx} = 0 \tag{2}$$

and the choices $\alpha \neq 0$, $\beta = 0$ and $s \neq 0$ lead equation (1) to the KdV equation

$$u_t + \alpha u u_x + s u_{xxx} = 0. \tag{3}$$

It is well known that both (2) and (3) are exactly solvable, and have the travelling wave solutions as follows:

$$u(x,t) = \frac{2k}{\alpha} + \frac{2\beta k}{\alpha}\tanh k(x-2kt)$$

and

$$u(x,t) = \frac{12sk^2}{\alpha}\operatorname{sech}^2 k(x-4sk^2t).$$

A great number of theoretical issues concerning the Burgers–KdV equation have received considerable attention. In particular, the travelling wave solution to the Burgers–KdV equation has been studied extensively. Johnson examined the travelling wave solutions to the Burgers–KdV equation in the phase plan by means of a perturbation method in the regimes where $\beta \ll s$ and $s \ll \beta$, and developed formal asymptotic expansions for the solution [7]. Grua and Hu used a steady-state version of (1) to describe a weak shock profile in plasmas [11]. They studied the same problem using a similar method to that used by Johnson [7], and a related problem was studied by Jeffrey [12]. The numerical investigation of the problem was carried out by Canosa and Gazdag *et al* [13–15]. Bona and Schonbek studied the existence and uniqueness of bounded travelling wave solutions to (1) which tend to constant states at plus and minus infinity [16]. They also considered the limiting behaviour of the travelling wave solution of (1) as $\beta \to 0$ with s of order 1, and also as $s \to 0$ with β of order 1. The case where both β and $s \to 0$ with β/s held fixed was also examined. The asymptotic and global behaviour of the travelling wave solution to (1) as $s > 0$ was undertaken by Guan and Gao, and the applications of the theory to diversified turbulent flow problems were described in detail in [9, 17]. On using variable transformation and the theory of ordinary differential equation, the asymptotic behaviour and the proper analytical solution to (1) were presented by Shu [18]. Gibbon *et al* showed that equation (1) does not have the Painlevé property [19]. Qualitative results concerning the travelling wave solutions to the Burgers–KdV equation were also obtained by the above mentioned authors and others, but they did not find the exact functional form of the travelling wave solution, or any other exact solutions.

Since the late 1980s, many mathematicians and physicists have obtained explicit exact solutions to the Burgers–KdV equation by various methods. Among them are Xiong, who obtained an exact solution to (1) when $\alpha = 1$, $\beta = -c$ and $s = \beta$ by the analytic method [20], Liu *et al* who obtained the same solution by the method of undetermined coefficients [21], Jeffrey *et al*, who obtained an exact solution to (1) by a direct method and a series method [22, 23], Halford and Vlieg-Hulstman, who obtained the same result in [24] by partial use of a Painlevé analysis, Wang, who applied the homogeneous balance method to obtain an exact solution [25], which was verified by Parkes by the tanh-function method [26]. However, apart from several minor errors in [25] and [26], the solutions obtained in the previous literature are actually equivalent to one another. That is, the travelling solitary wave solution to (1) can be expressed as a composition of a bell solitary wave and a kink solitary wave.

The purpose of this paper is to propose a new approach to the Burgers–KdV equation by using the theory of commutative algebra, which is currently called the first-integral method. The results obtained by this technique coincide with those presented in the previous literature.

2. Exact solutions to the Burgers–KdV equation

Assume that equation (1) has travelling wave solutions in the form $u(x,t) = u(\xi)$, $\xi = x - vt$, $(v \in \mathbb{R})$. Substituting it into equation (1) and integrating once we have

$$u''(\xi) - ru'(\xi) - au^2(\xi) - bu(\xi) - d = 0 \tag{4}$$

where $r = -\frac{\beta}{s}$, $a = -\frac{\alpha}{2s}$, $b = \frac{v}{s}$ and d is an arbitrary integration constant. Equation (4) is a nonlinear ordinary differential equation. It is commonly believed that it is very difficult for us to find exact solutions to equation (4) by usual ways [17]. Let $x = u$, $y = u_\xi$, then equation (4) is equivalent to

$$\begin{cases} \dot{x} = y \\ \dot{y} = ry + ax^2 + bx + d. \end{cases} \tag{5}$$

By the qualitative theory of ordinary differential equations [27], if we can find two first-integrals to (5) under the same conditions, then the general solutions to (5) can be expressed explicitly. However, in general, it is really difficult for us to realize this, even for one first-integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first-integrals, nor is there a logical way to tell us what these first-integrals are.

In this section, we are applying the first-integral method to study (1). That is, we will apply the Hilbert–Nullstellensatz theorem to obtain one first-integral to (5) which reduces equation (4) to a first-order integrable ordinary differential equation. An exact solution to (1) is then obtained by solving this equation. At the end of this section, the solutions obtained in the previous literature are compared with ours.

For convenience, first let us recall the Hilbert–Nullstellensatz theorem [28].

Hilbert–Nullstellensatz theorem

Let k be a field and L an algebraic closure of k.

(i) Every ideal γ of $k[X_1, \ldots, X_n]$ not containing 1 admits at least one zero in L^n.
(ii) Let $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n)$ be two elements of L^n; for the set of polynomials of $k[X_1, \ldots, X_n]$ zero at $\boldsymbol{x}$ to be identical with the set of polynomials of $k[X_1, \ldots, X_n]$ zero at $\boldsymbol{y}$, it is necessary and sufficient that there exists a k-automorphism s of L such that $y_i = s(x_i)$ for $1 \leqslant i \leqslant n$.
(iii) For an ideal α of $k[X_1, \ldots, X_n]$ to be maximal, it is necessary and sufficient that there exists an $\boldsymbol{x}$ in L^n such that α is the set of polynomials of $k[X_1, \ldots, X_n]$ zero at $\boldsymbol{x}$.
(iv) For a polynomial Q of $k[X_1, \ldots, X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $k[X_1, \ldots, X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.

Following immediately from the Hilbert–Nullstellensatz theorem, we obtain the division theorem for two variables in the complex domain $\mathbb{C}$:

Division theorem

Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, and $P(\omega, z)$ is irreducible in $\mathbb{C}[\omega, z]$. If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathbb{C}[\omega, z]$ such that

$$Q(\omega, z) = P(\omega, z) \cdot G(\omega, z).$$

Now, we apply the division theorem to seek the first-integral to (5). Suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to (5), and $p(x, y) = \sum_{i=0}^{m} a_i(x) y^i$ is an irreducible polynomial in $\mathbb{C}[x, y]$ such that

$$p[x(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(x) y^i = 0 \tag{6}$$

where $a_i(x)$ $(i = 0, 1, \ldots, m)$ are polynomials of x and all relatively prime in $\mathbb{C}[x, y]$, and $a_m(x) \not\equiv 0$. Equation (6) is also called the first-integral to (5). We start our study by assuming $m = 2$ in (6). Note that $\frac{\mathrm{d}p}{\mathrm{d}\xi}$ is a polynomial in x and y, and $p[x(\xi), y(\xi)] = 0$ implies $\frac{\mathrm{d}p}{\mathrm{d}\xi}\big|_{(5)} = 0$. By the division theorem, there exists a polynomial $H(x, y) = \alpha(x) + \beta(x)y$ in $\mathbb{C}[x, y]$ such that

$$\begin{aligned}
\frac{\mathrm{d}p}{\mathrm{d}\xi}\bigg|_{(5)} &= \left(\frac{\partial p}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \xi}\right)\bigg|_{(5)} \\
&= \sum_{i=0}^{2}\left[a_i'(x)y^i \cdot y\right] + \sum_{i=0}^{2}\left[\mathrm{i}a_i(x)y^{i-1}(ry + ax^2 + bx + d)\right] \\
&= [\alpha(x) + \beta(x)y]\left[\sum_{i=0}^{2} a_i(x)y^i\right].
\end{aligned} \tag{7}$$

On equating the coefficients of y^i $(i = 3, 2, 1, 0)$ on both sides of (7), we have

$$\mathbf{a}'(x) = \mathbf{A}(x) \cdot \mathbf{a}(x) \tag{8}$$

and

$$[0, ax^2 + bx + d, -\alpha(x)] \cdot \mathbf{a}(x) = 0 \tag{9}$$

where $\mathbf{a}(x) = (a_2(x), a_1(x), a_0(x))^t$, and

$$\mathbf{A}(\boldsymbol{x}) = \begin{pmatrix} \beta(x) & 0 & 0 \\ \alpha(x) - 2r & \beta(x) & 0 \\ -2(ax^2 + bx + d) & \alpha(x) - r & \beta(x) \end{pmatrix}.$$

Since $a_i(x)$ $(i = 0, 1, 2)$ are polynomials, from (8), we deduce that $a_2(x)$ is a constant and $\beta(x) = 0$. For simplification, taking $a_2(x) = 1$ and solving (8), we have

$$\mathbf{a}(\boldsymbol{x}) = \begin{pmatrix} 1 \\ \int[\alpha(x) - 2r]\,\mathrm{d}x \\ \int[a_1(x)\alpha(x) - ra_1(x) - 2(ax^2 + bx + d)]\,\mathrm{d}x \end{pmatrix}. \tag{10}$$

By (9) and (10), we conclude that $\deg \alpha(x) = 0$, i.e., $\deg a_1(x) = 1$. Otherwise, if $\deg \alpha(x) = k > 0$, then we deduce $\deg a_1(x) = k + 1$ and $\deg a_0(x) = 2k + 2$ from (10). This yields a contradiction with (9), since the degree of the polynomial $a_1(x) \cdot (ax^2 + bx + d)$ is $k + 3$, but the degree of the polynomial $a_0(x) \cdot \alpha(x)$ is $3k + 2$.

Assume that $a_1(x) = A_1 x + A_0$, $A_1, A_0 \in \mathbb{C}$ with $A_1 \neq 0$. By (10), we deduce that $A_1 = \alpha(x) - 2r$ and $a_0(x) = -\frac{2ax^3}{3} - bx^2 + \frac{A_1(A_1+r)}{2}x^2 - 2dx + A_0(A_1 + r)x + D$, here D is an arbitrary integration constant. Substituting $a_1(x)$ and $a_0(x)$ into (9) and setting all coefficients of x^i $(i = 3, 2, 1, 0)$ to zero we set

$$\begin{cases} A_1 a = \left(-\frac{2a}{3}\right) \cdot (A_1 + 2r) \\ A_0 a + A_1 b = \left[\frac{A_1(A_1+r)}{2} - b\right] \cdot (A_1 + 2r) \\ A_1 d + A_0 b = [(A_1 + r)A_0 - 2d] \cdot (A_1 + 2r) \\ A_0 d = D \cdot (A_1 + 2r). \end{cases} \tag{11}$$

Taking the integration constant $d = 0$, we have

$$A_1 = -\frac{4r}{5} \qquad A_0 = -\frac{12r^3}{125a} - \frac{2br}{5a} \qquad D = 0. \tag{12}$$

By the third equation of (11), we obtain

$$b = \frac{6r^2}{25} \qquad \text{or} \qquad b = -\frac{6r^2}{25}. \tag{13}$$

In the case $b = \frac{6r^2}{25}$, A_0 in (12) can be simplified as $A_0 = -\frac{4br}{5a}$. Substituting $a_0(x)$ and $a_1(x)$ into (6) we set

$$y^2 - \left(\frac{4r}{5}x + \frac{4br}{5a}\right)y - \frac{2a}{3}x^3 - bx^2 - \frac{2r^2}{25}x^2 - \frac{4br^2}{25a}x = 0. \tag{14}$$

From (14), y can be expressed in terms of x, i.e.

$$y = \frac{2r}{5}x + \frac{2br}{5a} \pm \sqrt{\frac{2a}{3}x^3 + 2bx^2 + \frac{2b^2}{a}x + \frac{2b^3}{3a^2}} = \frac{2r}{5}x + \frac{2br}{5a} \pm \sqrt{\frac{2}{3a^2}(ax+b)^3}. \tag{15}$$

Combining (5) and (15), we have

$$\frac{\mathrm{d}x}{\frac{2r}{5}x + \frac{2br}{5a} \pm (ax+b)\sqrt{\frac{2}{3a^2}(ax+b)}} = \mathrm{d}\xi. \tag{16}$$

By a transformation $z = \sqrt{\frac{2}{3a^2}(ax+b)}$, an exact solution to (1) can be obtained as follows by solving (16) directly

$$u(x,t) = \frac{3a}{2}\left[\frac{\pm\frac{2r}{5a}\mathrm{e}^{\frac{r}{5}\xi}}{\mathrm{e}^{\frac{r}{5}\xi} + c}\right]^2 - \frac{b}{a} = -\frac{12\beta^2}{25\alpha s}\left[\frac{\mathrm{e}^{-\frac{\beta}{5s}\xi}}{\mathrm{e}^{-\frac{\beta}{5s}\xi} + c}\right]^2 + \frac{2v}{\alpha} \tag{17}$$

where $\xi = x - vt$ and c is an arbitrary integration constant.

Since $b = \frac{v}{s}$ and $r = -\frac{\beta}{s}$, $b = \frac{6r^2}{25}$ in (13) implies $v = \frac{6\beta^2}{25s}$. By using the equality $4A\left[\frac{\mathrm{e}^{2t}}{1+\mathrm{e}^{2t}}\right]^2 = -A\ \mathrm{sech}^2 t + 2A\tanh t + 2A$ and setting $c = 1$ in (17), the explicit travelling solitary wave solution to equation (1) can be rewritten as

$$u(x,t) = \frac{3\beta^2}{25\alpha s}\mathrm{sech}^2\left[\frac{1}{2}\left(-\frac{\beta}{5s}x + \frac{6\beta^3}{125s^2}t\right)\right] - \frac{6\beta^2}{25\alpha s}\tanh\left[\frac{1}{2}\left(-\frac{\beta}{5s}x + \frac{6\beta^3}{125s^2}t\right)\right] + \frac{6\beta^2}{25\alpha s}. \tag{18}$$

Similarly, in case $b = -\frac{6r^2}{25}$, an exact solution to (1) is as follows:

$$u(x,t) = -\frac{12\beta^2}{25\alpha s}\left[\frac{\mathrm{e}^{-\frac{\beta}{5s}\xi}}{\mathrm{e}^{-\frac{\beta}{5s}\xi} + c}\right]^2 \tag{19}$$

where $\xi = x - vt$ and c is an arbitrary integration constant.

Note that $b = -\frac{6r^2}{25}$ in (13) implies $v = -\frac{6\beta^2}{25s}$. By setting $c = 1$ in (19), explicit travelling solitary wave solutions to equation (1) can be rewritten as

$$\begin{aligned} u(x,t) = {}& \frac{3\beta^2}{25\alpha s}\mathrm{sech}^2\left[\frac{1}{2}\left(-\frac{\beta}{5s}x - \frac{6\beta^3}{125s^2}t\right)\right] \\ & - \frac{6\beta^2}{25\alpha s}\tanh\left[\frac{1}{2}\left(-\frac{\beta}{5s}x - \frac{6\beta^3}{125s^2}t\right)\right] - \frac{6\beta^2}{25\alpha s}. \end{aligned} \tag{20}$$

Equations (17) and (19) also confirm the qualitative analysis of equation (1) by Guan and Gao in [17].

If assuming $m = 3, 4$ in (6), respectively, using similar arguments to those given earlier, we obtain that (5) does not have any first-integral in the form (6). We do not need to discuss the cases $m \geqslant 5$ due to the fact that, in general, the polynomial equation with degree greater or equal to 5 is not solvable.

Now let us compare (17) and (19) with the solutions obtained in the previous literature. Xiong *et al* used the analytic method and the method of undetermined coefficients to study the Burgers–KdV equation

$$u_t + uu_x - cu_{xx} + \beta u_{xxx} = 0. \tag{21}$$

With the assumption $6c^2 = 25\beta v$, an exact solution to equation (21) is presented in [20, 21] as

$$u(\xi) = 2v - \frac{3c^2}{25\beta}\left(1 + \tanh\frac{c\xi}{10\beta}\right)^2. \tag{22}$$

By using the equality $\frac{Ae^t}{1+e^t} = \frac{A}{2}\tanh\left(\frac{t}{2}\right) + \frac{A}{2}$, it is easy to see that (22) is in agreement with (18) as $\alpha = 1$, $\beta = -c$ and $s = \beta$.

By means of a direct method and a series method, Jeffrey *et al* investigated the Burgers–KdV equation in [22, 23]

$$u_t + 2auu_x + 5bu_{xx} + cu_{xxx} = 0$$

and found the solutions as follows provided that (2.9a, b) and (2.10a, b) hold on p 561 in [22]

$$u_1 = \frac{3b^2}{2ac}\left[\operatorname{sech}^2\left(\frac{\theta}{2}\right) + 2\tanh\left(\frac{\theta}{2}\right) + 2\right] \tag{23}$$

where $\theta = \frac{b}{c}x - \frac{6b^3}{c^2}t + d$, and

$$u_2 = \frac{3b^2}{2ac}\left[\operatorname{sech}^2\left(\frac{\theta}{2}\right) - 2\tanh\left(\frac{\theta}{2}\right) - 2\right] \tag{24}$$

where $\theta = -\frac{b}{c}x - \frac{6b^3}{c^2}t + d$. Since $\tanh(-x) = -\tanh(x)$, it is easily seen that (23) and (24) are identical with (18) and (20) respectively as $\alpha = 2a$, $\beta = 5b$, and $s = c$, and the conditions (2.9a, b) and (2.10a, b) agree with (13). By partial use of a Painlevé analysis, the same results are presented by Halford and Vlieg-Hulstman in [24].

Taking $s = -s$, $\alpha = p$ and $\beta = r$ in equation (1), Wang applied the homogeneous balance method to equation (1) in [25]. The corresponding result ((3.18) when taking $b = 0$) in [25] appears to coincide with (18) and (20). Note that there are three errors on p 286 in [25]: one is that the first term in (3.16) should be $-\frac{6r^3}{125s^2}$, as a result of which the denominator $125s$ in (3.18) should be replaced by $125s^2$. Parkes and Duffy also studied the Burgers–CKdV and Burgers–KdV equations by the tanh-function method, but there are also some similar errors on p 219 in [26].

3. Conclusion

The methods used in [20–24] are very useful for nonlinear evolution equations. The method of solution of (1) used in [25] is complicated and not efficient. The method for (1) used in [26] is only limited to the equations which have the travelling wave solution in the form of a power of series in tanh-function. The first-integral method described herein is not only efficient but also has the merit of being widely applicable. We can definitely apply this technique to many nonlinear evolution equations, such as the nonlinear Schrödinger equation, the generalized Klein–Gordon equation, and the higher order KdV-like equation, which can be converted to the following form through the travelling wave transformation

$$u''(\xi) - \mu u'(\xi) - R(u) = 0$$

where μ is real and $R(u)$ is a polynomial with real coefficients. We believe that this method must be advantageous for a rather diverse group of scientists.

Acknowledgment

Part of the content was presented in 'Sixth SIAM Conference on Applications of Dynamical Systems', Snowbird, Utah, USA, May 2001. The work has been presented in Applied Mathematics Seminar, Department of Mathematics, Texas A&M University, TX, September 2001. The author would like to thank Professor Goong Chen for his useful suggestions.

References

[1] Toda M 1970 *Prog. Theor. Phys. Suppl.* **45** 174
[2] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Springer)
[3] Kruskal M D 1974 *The Korteweg–de Vries Equation and Related Evolution Equations* (Providence, RI: American Mathematical Society)
[4] Love M D 1980 *J. Fluid Mech.* **100** 87
[5] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
[6] Medina E, Hwa T, Kardar M and Zhang Y C 1989 *Phys. Rev.* A **39** 3053
[7] Johnson R S 1970 *J. Fluid Mech.* **42** 49
[8] van Wijngaarden L 1972 *Ann. Rev. Fluid Mech.* **4** 369
[9] Gao G 1985 *Sci. Sin.* A **28** 616
[10] Liu S D and Liu S K 1992 *Sci. Sin.* A **35** 576
[11] Grua H and Hu P W 1967 *Phys. Fluids* **10** 2596
[12] Jeffrey A 1979 *Arch. Mech.* **31** 559
[13] Canosa J and Gazdag J 1977 *J. Comput. Phys.* **23** 393
[14] Dauletiyarov K Z 1984 *Zh. Vychisl. Mat. Mat. Fiz.* **24** 383
[15] Turetaev I D 1991 *Comput. Math. Math. Phys.* **31** 69
[16] Bona J L and Schonbek M E 1985 *Proc. R. Soc. Edin.* **101** 207
[17] Guan K Y and Gao G 1987 *Sci. Sin.* A **30** 64
[18] Shu J J 1987 *J. Phys. A: Math. Gen.* **20** L49
[19] Gibbon J D, Radmore P, Tabor M and Wood D 1985 *Stud. Appl. Math.* **72** 39
[20] Xiong S L 1989 *Chin. Sci. Bull.* **34** 1158
[21] Liu S D, Liu S K and Ye Q X 1998 *Math. Prac. Theory* **28** 289
[22] Jeffrey A and Xu S 1989 *Wave Motion* **11** 559
[23] Jeffrey A and Mohamad M N B 1991 *Wave Motion* **14** 369
[24] Halford W D and Vlieg-Hulstman M 1991 *Wave Motion* **14** 267
[25] Wang M 1996 *Phys. Lett.* A **213** 279
[26] Parkes E J and Duffy B R 1997 *Phys. Lett.* A **229** 217
[27] Ding T R and Li C Z 1996 *Ordinary Differential Equations* (Peking: Peking University Press)
[28] Bourbaki N 1972 *Commutative Algebra* (Paris: Addison-Wesley)